Introduction to Brauer Manin Obstruction.

§ Motivation.

Q: Given $y^2 = 3x^3 + 2$.

Find a rational solution for the equation.

Idea:

Try $x = \frac{a}{b}$ $y = \frac{c}{d}$ such that $\max\{|a|, |b|, |c|, |d|\} < B$ ← a constant.

keep searching by enlarge B.

"height"

If there is such rational solution, we can find one by this method.

If there is no rational solution, we can't determine it by searching.

Q: How to determine solution set is empty?

More generally.

for a ring R. we want to determine when the scheme

$$X = V(f_1, f_2 \cdots f_r) = \{(a_1, a_2 \cdots a_n) \in R^n \mid f_1 = f_2 = \cdots = f_n = 0\}$$

has a solution in R^n.

where f_i is a polynomial with coefficient R.

i.e. $f_i \in R[x_1, \cdots x_n]$

Notation $X(R) = \{R \text{ solutions of } X\}$

$= \{ R\text{-valued points of } X \}$

$= \mathrm{Hom}(\mathrm{Spec} R, X)$.

$= \mathrm{Hom}\left(\cancel{\dfrac{R[x_1 \cdots x_n]}{(f_1 \cdots f_n)}}, R \right)$

☆ Assume X is a finite type scheme over ring R for the whole talk.

Ring R	$\exists$ an algorithm to determine if $X(R) = \phi$
$\mathbb{C}$	Yes
$\mathbb{R}$	Yes
$\mathbb{F}_p$	Yes
$\mathbb{Q}_p$	Yes
$\mathbb{Q}$	?
$\mathbb{F}_p(t)$	No
$\mathbb{Z}$	No.

What is $\mathbb{Q}_p$ (local field)?

$y^2 = 3x^3 + 2 \mod p$, solution in $\mathbb{Z}/p\mathbb{Z}$

$y^2 = 3x^3 + 2 \mod p^2$ in $\mathbb{Z}/p^2\mathbb{Z}$

$y^2 = 3x^3 + 2 \mod p^3$ in $\mathbb{Z}/p^3\mathbb{Z}$

Take inverse limit. $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$

$\mathbb{Q}_p$ is the fractional field of integral domain $\mathbb{Z}_p$

or. ($\mathbb{Q}_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q}$.)

By construction, we know

$$X(\mathbb{Q}) \neq \phi \Rightarrow X(\mathbb{Q}_p) \neq \phi$$

Generally. Given a number field k. and a prime v

we can construct a local field k_v

Similarly. $X(k_v) = \phi \Rightarrow X(k) = \phi$

Now we know

$$X(k) \subset X(k_v)$$

we have algorithm to determine if $X(k_v) = 0$.

So $X(K) \subset \prod_v X(k_v)$

Right hand side is a product of infinite term. It seems to be not computable anymore. We take a subset of it.

Adelic point of X: $X(\mathbb{A}_K) \subset \prod_v X(k_v)$

computable.

$$\mathbb{A}_k := \prod_v{}' (k_v, \mathcal{O}_v) = \{(a_v) \in \prod_v k_v \mid \#\{v \mid a_v \notin \mathcal{O}_v\} < \infty\}$$

If X is smooth projective geometrically integral variety

then $X(\mathbb{A}_K) = \prod_v{}' (X(K_v), X(\mathcal{O}_v))$

Now $X(K) \hookrightarrow X(\mathbb{A}_K)$ ← we have algorithm to determine $X(\mathbb{A}_K) = \emptyset$

Q: what if $X(\mathbb{A}_K) \neq \emptyset$ how can we determine if $X(k) = \emptyset$?

Idea:

Find a set T such that $X(K) \subset T \subset X(\mathbb{A}_K)$

such that $\exists$ algorithm to determine if $T = \emptyset$

Even if $X(\mathbb{A}_K) \neq \emptyset$, if we determine $T = \emptyset$, then we can also determine $X(k) = \emptyset$

T is called the obstruction

Thm: $X(k) \subset X(\mathbb{A}_k)^{Br} \subset X(\mathbb{A}_k)$

$X(\mathbb{A}_k)^{Br}$ is called Brauer-Manin obstruction.

§ Brauer groups

$$\begin{aligned} Br(k) &= H^2_{et}(\operatorname{Spec} k, \mathbb{G}_m) \\ &= H^2(G_k, (k^{sep})^\times) \quad \text{where } G_k = Gal(k^{sep}/k) \\ &= \{\text{central simple algebras over } k\} / \sim \end{aligned}$$

Let A be a finite dimensional k-algebra

A is called simple if A doesn't have non trivial two side idea.

A is called central if center of A is k.

Central simple algebra of A over k if A is finite dimensional. k algebra that is central and simple.

Ex. Matrix algebra $M_n(k)$ is central simple algebra.

Ex. Halmilton's quaternion algebra $\mathbb{H}$ over $\mathbb{R}$ is central simple algebra.

$\{$ Central simple algebra over $k\}$

Define equivalent relation

$$A \sim B \text{ if } A \otimes_k M_n(k) \simeq B \otimes_k M_m(k) \text{ for some integer } m\ n.$$

Define Multiplication.

$$[A] \times [B] = [A \otimes_k B]$$

A^{opp} is the opposite algebra of A. i.e. $\begin{cases} A^{op} = A \text{ as a set (vector space)} \\ \text{mulplication in } A^{op}: x \times y = yx \end{cases}$

$$A \otimes A^{opp} = M_n(k)$$

$\boxed{\text{Abelian}}$

So $[A]$ has inverse.

$Br(k) = \{$ central simple algebra over $k\} / \sim$ is a Abelian group.

Brauer groups of schemes.

Def: $Br(X) = H^2_{et}(X, \mathbb{G}_m)$

Remark: it is also possible to generalize the third definition.

Def: An Azumaya algebra is a coherent $\mathcal{O}_X$-algebra A such that for any point $x \in X$, the fiber $A \otimes_{\mathcal{O}_X} k(x)$ is a central simple algebra over the residue field $k(x)$

Two Azumaya algebra are similar to each other if there are locally free sheaf $\mathcal{E}_1$ and $\mathcal{E}_2$ such that

$$A_1 \otimes_{\mathcal{O}_X} \mathrm{End}\,\mathcal{E}_1 \simeq A_2 \otimes_{\mathcal{O}_X} \mathrm{End}\,\mathcal{E}_2$$

The similarity class of Azumaya algebra forms a group.
If X is regular and quasi-projective. then the group is iso to $Br(X)$

§ Brauer - Manin obstruction

- $Br(-) = H^2_{et}(-, \mathbb{G}_m)$ is a contravariant functor

i.e. $f: X \longrightarrow Y$ morphism of scheme,
then it induced $f^*: Br(Y) \longrightarrow Br(X)$

$$\begin{array}{ccc} X(k) & \hookrightarrow & X(k_v) \\ \| & & \| \\ \mathrm{Hom}(\mathrm{Spec}\, k, X) & & \mathrm{Hom}(\mathrm{Spec}\, k_v, X) \\ \alpha & \longrightarrow & \alpha_v \end{array}$$

$$\begin{tikzcd} \operatorname{Spec} k_v \arrow[r] \arrow[rr, bend right, "\alpha_v"'] & \operatorname{Spec} k \arrow[r, "\alpha"] & X \end{tikzcd}$$

$$\begin{tikzcd} Br(X) \arrow[r, "\alpha^*"] \arrow[rr, bend right, "\alpha_v^*"'] & Br(k) \arrow[r] & Br(k_v) \end{tikzcd}$$

In other word.

Let $A \in Br(X)$

$$\begin{tikzcd} X(k) \arrow[r, hook] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\circlearrowleft"] & X(k_v) \arrow[d, "\varphi_{v,A}"] \\ Br(k) \arrow[r] & Br(k_v) \end{tikzcd}$$

where $\varphi_A(\alpha) = \alpha^* A$

$\varphi_{v,A}(\alpha_v) = \alpha_v^* A$

Hence
$$\begin{tikzcd} X(k) \arrow[r, hook] \arrow[d, "\varphi_A"'] & \prod X(k_v) \arrow[d, "\prod \varphi_{v,A}"] \\ Br(k) \arrow[r] & \prod Br(k_v) \end{tikzcd}$$

Recall, we take a subspace $X(A_k) \subset \prod X(k_v)$

It fit into such a commutative diagram.

$$\begin{tikzcd} X(k) \arrow[r, hook] \arrow[d, "\varphi_A"'] & X(k_v) \arrow[d, "\gamma_A"] \\ Br(k) \arrow[r] & \bigoplus_v Br(k_v) \end{tikzcd}$$

By class field theory

$$\begin{tikzcd} & X(k) \arrow[r, hook, "i"] \arrow[d, "\varphi_A"'] & X(\mathbb{A}_k) \arrow[d, "\gamma_A"] & & \\ 0 \arrow[r] & Br(k) \arrow[r, "j"] & \bigoplus_v Br(k_v) \arrow[r, "\Sigma"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

$(Br(k_v) \simeq \mathbb{Q}/\mathbb{Z})$

For $\alpha \in X(k)$

$$\Sigma \circ \gamma_A \circ i(A) = \Sigma \circ j \circ \varphi_A(A) = 0$$

Hence $\alpha \in \{ \alpha \in X(\mathbb{A}_k) \mid \Sigma \circ \gamma_A(\alpha) = 0 \} =: X(\mathbb{A}_k)^A$.

$$X(k) \subset X(\mathbb{A}_k)^A$$

i.e. $X(k) \subset \bigcap_{A \in Br(X)} X(\mathbb{A}_k)^A$.

Def: Brauer Manin obstruction $X(\mathbb{A}_k)^{Br} := \bigcap_{A \in Br(X)} X(\mathbb{A}_k)^A$

We can also think it in this way.

$$\begin{aligned} X(\mathbb{A}_k) \times Br(X) &\longrightarrow \mathbb{Q}/\mathbb{Z} \\ (\alpha, \quad A) &\longmapsto \Sigma \circ \gamma_A(\alpha) \end{aligned}$$

$X(\mathbb{A}_k)^A$ consists of elements in $X(\mathbb{A}_k)$ that is orthogonal to A.

$X(\mathbb{A}_k)^{Br}$ consists of elements in $X(\mathbb{A}_k)$ orthogonal to $Br(X)$

$$X(k) \subset X(\mathbb{A}_k)^{Br} \subset X(\mathbb{A}_k)$$

Q: Is it useful.

In theory: Yes. There are many examples such that

$$X(\mathbb{A}_k) \neq \phi, \quad X(\mathbb{A}_k)^{Br} = \phi$$

We can use Brauer Manin obstruction to argue that $X(k)=0$

In fact: typically expect $X(\mathbb{A}_k)^A \subsetneq X(\mathbb{A}_k)$ "unless forced otherwise"

In practice: Not quite Yes.. Can be computed in several examples but no generereral effective way to compute it.

(Assume X is a smooth projective variety over number field K. now) can skip

Fact: If $X(\mathbb{A}_k)^{Br} = \phi$, there exists a finite set $B \subset Br(X)$ such that $X(\mathbb{A}_k)^B = \phi$

idea: Find subgp $B \subset Br(X)$ such that $X(\mathbb{A}_k)^B$ is computable

Def. We say B captures the Brauer-Manin obstruction if $X(\mathbb{A}_k)^{Br} = \phi \Rightarrow X(\mathbb{A}_k)^B = \phi$

Theorem ① If C is a smooth projective degree d genus 1 curve, then $Br(C[d^\infty])$ (completely) captures the Brauer-Manin obstruction.

② If X is a smooth projective cubic obstruction. then $Br\, X[3]$ (completely) captures the Brauer-Manin Obstruction.

For most example in literature, they show $X(\mathbb{A}^k)^{Br} = \phi$ by show $X(\mathbb{A}^k)^A = 0$ for only one element $A \in Br(X)$

But generally, we need a lot elements in $Br(X)$.

Fact: Let $A_1, A_2 \in Br(X)$. then

$$X(\mathbb{A}_k)^{A_1} \cap X(\mathbb{A}_k)^{A_2} = \bigcap_{\gamma \in \{A_1^i A_2^j \mid i, j \in \mathbb{Z}\}} X(\mathbb{A}_k)^{\gamma}$$

can skip

We just need finite the generators A_i of $Br(X)$ and calculate $Br(X)^{A_i}$, then take intersection

Thm. Let $N \geq 0$, $char(k) \neq 2$. ← number field, we don't care.

$\exists$ smooth projective geometrically integral variety X over global field k s.t. $X(\mathbb{A}_k)^{Br} = \phi$ but $\forall$ subgp $B \subset Br(X)$ generated by $< N$ elements, $X(\mathbb{A}_k)^B \neq \phi$.

§ Structure of $Br(X)$

Let $\pi: X \to Spec\, k$ be the structure morphism.

$Br_0(X) = Im(\pi^*: Br(k) - Br(X))$ "constant Brauer classes"

Fact: $X(\mathbb{A}_k)^{Br_0(X)} = X(\mathbb{A}_k)$

$X(\mathbb{A}_k)^{Br}$ only depends on $Br(X) / Br_0(X)$

Hochschild-serre spectral sequence to Galois cover $\bar{X} \to X$ and sheaf $\mathbb{G}_m$

$$H^p(G_k, H^q_{et}(\bar{X}, \mathbb{G}_m)) \Rightarrow H^{p+q}_{et}(X, \mathbb{G}_m)$$

exact sequence of low degree terms

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \bar{X})^{G_K} \to \operatorname{Br} k \to \underbrace{\ker(\operatorname{Br}(X) \to \operatorname{Br}(\bar{X}))}_{=\operatorname{Br}_1(X)}$$

$$\to H^1(G_K, \operatorname{Pic} \bar{X}) \to \underbrace{H^3(G_K, \bar{k}^\times)}_{=0}$$

for number field.

$$0 \to \operatorname{Br}_0(K) \to \operatorname{Br}_1(K) := \ker(\operatorname{Br}(X) \to \operatorname{Br}(\bar{X})) \to H^1(G_K, \operatorname{Pic} \bar{X}) \to 0$$

The Hochschild-serre spectral seq also gives.

$$0 \to \frac{\operatorname{Br}(X)}{\operatorname{Br}_1(X)} \to (\operatorname{Br}(\bar{X}))^{G_K} \to H^2(G_K, \operatorname{Pic} \bar{X})$$

If we have a good understanding of $\operatorname{Br}(\bar{X})$ and $\operatorname{Pic} \bar{X}$ as Galois modues, we can compute $\operatorname{Br}(X)/\operatorname{Br}_0(X)$.

Reference:

Rational points on varieties and the Brauer-Manin obstruction
— Bianca Viray

The Brauer-Manin obstruction
— Shelly Manber